SB-3535

M. A. / M. Sc. (Part-II) Examination

March / April – 2011

Mathematics : Paper - 5007

(Special Functions)

(New Course)

Time : Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.

Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :

☛ **M. A. / M. Sc. (Part-2)**

Name of the Subject :

☛ **Mathematics - 5007**

☛ Subject Code No. : **3 5 3 5** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) Attempt **all** questions.

(3) Figures to the right indicates marks.

(4) Notations and conventions are all standard.

1 (a) Show that the product $\prod_{n=1}^{\infty}(1+a_n)$ with zero factors deleted is absolutely convergent iff $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. **5**

(b) Show that if z is not a negative integer, then **5**

$$\lim_{n \to \infty} \frac{(n-1)!\, n^z}{(z+1)(z+2)\cdots(2+n-1)}$$ exists.

(c) Show that the product $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ is convergent and find its value. **4**

OR

1 (a) If $a_n \neq -1$, prove that $\prod_{n=1}^{\infty}(1+a_n)$ and $\sum_{n=1}^{\infty}\log(1+a_n)$ converges or diverges together. **5**

(b) Define Legendre polynomials. Obtain the values of $P_{2n+1}(0)$ and $P_{2n}(0)$ **5**

(c) Show that $\log(1+x) = x\,F(1, 1; 2\,;\,x)$ **4**

2 (a) If $|z|<1$ and $|1-z|<1$, $\text{Re}(c)>1$, $\text{Re}(c-a-b)>0$ and if non of a, b, $c-a, c-b$, $c-a-b$ is an integer then prove that $F(a, b; a+b+1-c; 1-z) = AF(a, b; c; z)$ $+BZ^{1-c}F(a+1-c, b+1-c; 2-c; z)$ where **8**

$$A = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(a+1-c)\,\Gamma(b+1-c)} \quad \text{and} \quad B = \frac{\Gamma(a+b+1-c)\,\Gamma(1-c)}{\Gamma(a)\,\Gamma(b)}$$

(b) If $|z|<1$ and $\left|\frac{z}{1-z}\right|<1$ then show that **6**

$$F(a, b; c; z) = (1-z)^{-a}\,F\left(a, c-b; c; \frac{-z}{1-z}\right)$$

OR

2 (a) With usual notation prove that **8**

$$F\begin{bmatrix} a, & b\,; \\ 1+a-b; & \end{bmatrix} -1 = \frac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{a}{2}\right)}{\Gamma\left(1+\frac{a}{2}-b\right)\,\Gamma(1+a)}$$

(b) Define contiguous function to $F(a, b; c; z)$. Derive the relations : **6**

(i) $(a-b)F = aF(a+) - bF(b+)$

(ii) $[a+(b-c)z]\,F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+)$

3 (a) For $t \neq 0$ and for all finite z show that **6**

$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)t^n$$

(b) Define the Bessel's function $J_n(z)$. Show that **4**

$J_n(-z) = (-1)^n J_n(z)$, $J_{-n}(z) = (-1)^n J_n(z)$.

(c) Show that $\frac{d^n}{dx^n}\left[x^{a-1+n}F(a, b; c; x)\right]$ **4**

$= (a)_n x^{a-1} F(a+n, b; c; x)$

OR

3 (a) Show that the Neumann polynomials are given by **6**

$O_0(s) = s^{-1}$ and $O_n(s) = \frac{n}{4}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(n-1-k)!}{k!}\left(\frac{2}{5}\right)^{n+1-2k}$, $n \geq 1$.

(b) If $a+b+\frac{1}{2}$ is neither zero nor negative integer and **5**

if $|x|<1$ and $|4x(1-x)|<1$ Prove that

$$F\left[\begin{matrix} a, b; \\ a+b+\frac{1}{2}; \end{matrix} \; 4x(1-x)\right] = F\left[\begin{matrix} 2a, 2b; \\ a+b+\frac{1}{2}; \end{matrix} \; x\right]$$

(c) Prove that $\sum_{n=0}^{\infty} J_{2n+1}(x) = \frac{1}{2}\int_0^x J_0(y)\,dy$ **3**

4 (a) If $-1 < x < 1$ and if n is any integer show that **5**

$$|p_n(x)| < \left[\frac{\pi}{2n\left(1-x^2\right)}\right]^{\frac{1}{2}}$$

(b) Prove that **5**

$$P_n(\cos\alpha) = \left(\frac{\sin\alpha}{\sin\beta}\right)^n \sum_{k=0}^{n} C_{n,k} \left\{\frac{\sin(\beta-\alpha)}{\sin\alpha}\right\}^{n-k} P_k(\cos\beta)$$

(c) With usual notation prove that **4**

$$\sum_{n=0}^{\infty} \frac{P_n(x)t^n}{n!} = e^{xt} J_0\left(t\sqrt{1-x^2}\right)$$

OR

4 (a) In usual notation prove that $\int_0^{\pi/2} P_n^2(x)\,dx = \frac{2}{2n+1}$ **5**

(b) For $-1 < x < 1$ show that $|P_n(x)| < 1$ **5**

(c) Show that $\int_{-1}^{1} (1+x)^{\alpha-1}(1-x)^{\beta-1} P_n(x)\,dx$ **4**

$$= 2^{\alpha+\beta-1} B(\alpha, \beta)\, {}_3F_2\left[\begin{matrix} -n,\ n+1; \\ 1,\ \alpha+\beta\ ; \end{matrix}\ 1\right]$$

5 (a) Define the Hermite polynomials. Derive the Rodrigues formula for $H_n(x)$. **6**

(b) Show that

$$H_n(x) = \frac{n!}{\pi}\int_0^{\pi} \exp(2x\cos\theta - \cos 2\theta)\cdot\cos(2x\sin\theta - \sin 2\theta - n\theta)\,d\theta$$ **4**

(c) Show that $H_{2n+1}(x) = (-1)^n 2^{2n+1} n!\, x L_n^{\left(\frac{1}{x}\right)}(x^2)$ **4**

OR

5 (a) With usual notation prove that **6**

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n^2(x)\,dx = 2^n n!\sqrt{x}$$

(b) Prove that $x^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!\, H_{n-2k}(x)}{2^n\ k!\,(n-2k)!}$ **4**

(c) Show that $H_n(x) = 2xH_{n-1}(x) - H'_{n-1}(x)$ **4**